Quadratic equation

$$ax^2 + bx + c = 0$$

Equation $ax^2 + bx + c = 0$, where x is unknown; a, b and c real numbers, $a \neq 0$, is quadratic equation "by x" with the coefficient a, b and c.

Quadratic equation **is complete**, if the coefficients $b \neq 0$ and $c \neq 0$.

If $b = 0$ or $c = 0$ (or both) then quadratic equation is **incomplete.**

Incomplete square equation is relatively easy to solve.

Incomplete square equations

$ax^2 + bx = 0$

$x(ax + b) = 0$

$x = 0 \quad \vee \quad ax + b = 0$

$x = -\frac{b}{a}$

$ax^2 + c = 0$

$ax^2 = -c$

$x^2 = -\frac{c}{a}$

$x = \pm\sqrt{-\frac{b}{a}}$

$ax^2 = 0$

$x = 0$

Examples:

$2x^2 + 5x = 0$

$x(2x + 5) = 0$

$x = 0 \quad \vee \quad 2x + 5 = 0$

$2x = -5$

$x = -\frac{5}{2}$

$4x^2 - 9 = 0$

$4x^2 = 9$

$x^2 = \frac{9}{4}$

$x = \pm\sqrt{\frac{9}{4}}$

$x = \pm\frac{3}{2}$

$x_1 = \frac{3}{2}$

$x_2 = -\frac{3}{2}$

$5x^2 = 0$

$x^2 = \frac{0}{5}$

$x = 0$

$x_1 = x_2 = 0$

Complete square equation: $ax^2+bx+c=0$

Quadratic equation has two solutions: x_1 and x_2.

$$\boxed{x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

Example 1. Solve the equations:

a) $6x^2-x-2=0$
b) $x^2-2x+1=0$
c) $x^2-4x+5=0$

Solution:

a) $6x^2-x-2=0$

$a=6$

$b=-1$

$c=-2$

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-1)\pm\sqrt{(-1)^2-4\cdot6\cdot(-2)}}{2\cdot6}$$

$$x_{1,2}=\frac{1\pm\sqrt{49}}{12}=\frac{1\pm17}{12}$$

$$x_1=\frac{1+7}{12}=\frac{8}{12}=\frac{2}{3}$$

$$x_2=\frac{1-7}{12}=\frac{-6}{12}=-\frac{1}{2}$$

b) $x^2-2x+1=0$

$a=1$

$b=-2$

$c=1$

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-2)\pm\sqrt{(-2)^2-4\cdot1\cdot1}}{2\cdot1}$$

$$x_{1,2}=\frac{2\pm\sqrt{4-4}}{2}=\frac{2\pm0}{2}$$

$$x_1=\frac{2}{2}=1$$

$$x_2=\frac{2}{2}=1$$

c) $x^2 - 4x + 5 = 0$

$a = 1$
$b = -4$
$c = 5$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{16 - 20}}{2 \cdot 1}$$

$$x_{1,2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = \frac{2(2 \pm i)}{2} = 2 \pm i$$

$x_1 = 2 + i$ Look out for: $\sqrt{-4} = \sqrt{4(-1)} = 2i$

$x_2 = 2 - i$ $\sqrt{-1} = i$

Example 2. Solve the equation: $(2x-3)^2 + (x-1)(x+2) = 2 - 11x$

Solution:

$(2x-3)^2 + (x-1)(x+2) = 2 - 11x$

$4x^2 - 12x + 9 + x^2 + 2x - x - 2 - 2 + 11x = 0$

$5x^2 + 5 = 0 / :5$

$x^2 + 1 = 0 \rightarrow$ Incomplete quadratic equation

$x^2 = -1$

$x = \pm\sqrt{-1}$

$x_1 = +i$

$x_2 = -i$

Example 3. Solve the equation: $\frac{x}{x-2} - \frac{3}{x+2} = \frac{8}{x^2 - 4}$

Solution:

$$\frac{x}{x-2} - \frac{3}{x+2} = \frac{8}{x^2 - 4}$$

$\frac{x}{x-2} - \frac{3}{x+2} = \frac{8}{(x-2)(x+2)} \rightarrow$ Multiply all with $S = (x-2)(x+2)$ $x \neq 2$ $x \neq -2$

$x(x+2) - 3(x-2) = 8$

$x^2 + 2x - 3x + 6 - 8 = 0$

$x^2 - x - 2 = 0 \rightarrow$ Now working as a quadratic equation

$a = 1$
$b = -1$
$c = -2$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-2)}}{2}$$

$$x_{1,2} = \frac{1 \pm 3}{2}$$

$x_1 = \frac{1+3}{2} = \frac{4}{2} = 2 \rightarrow$ **Watch out: it is not** a solution because $x \neq 2$

$x_2 = \frac{1-3}{2} = \frac{-2}{2} = -1 \rightarrow$ So $x = -1$ is only solution!

Nature of solutions

$ax^2 + bx + c = 0 \longrightarrow D = b^2 - 4ac$

Now , solutions, we can write as: $x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$

For quadratic equation $ax^2 + bx + c$ with the real coefficient is:

1) Equation has two different real solutions if and only if $D > 0$

$(x_1 = x_2 \in R \;\; x_1 \neq x_2$ **iff** $D > 0)$

2) Equation has a double real solution if and only if $D = 0$

$(x_1 = x_2 \in R$ **iff** $D = 0)$

3) Equation has a pair of complex solutions if and only if $D < 0$

$(x_1 = a + bi, x_2 = a - bi$ **iff** $D < 0)$

Example 1. Identify the nature of solutions in square equation, depending on the parameters:

a) $x^2 + 3x + m = 0$
b) $(n+3)x^2 - 2(n+1)x + n - 5 = 0$

Solution:

a) $x^2+3x+m=0$ ⟶ $a=1$, $b=3$, $c=m$

$D=b^2-4ac=3^2-4\cdot1\cdot m=9-4m$

1) $D>0 \;\Rightarrow\; 9-4m>0$

$-4m>-9 \rightarrow$ **Watch out:** turning the sign

$m<\frac{-9}{-4}$

$m<\frac{9}{4}$

2) $D=0 \;\Rightarrow\; 9-4m=0 \;\Rightarrow\; m=\frac{9}{4}$

3) $D<0 \;\Rightarrow\; 9-4m<0 \;\Rightarrow\; m>\frac{9}{4}$

So: - For $m<\frac{9}{4}$ solutions are real and different

- For $m=\frac{9}{4}$ solutions are real and equal

-For $m>\frac{9}{4}$ solutions are complex numbers

b) $(n+3)x^2-2(n+1)x+n-5=0$

$a=n+3$

$b=-2(n+1)$ **Watch out:** $n+3\neq0$

$c=n-5$

$D=b^2-4ac=\left[-2(n+1)\right]^2-4(n+3)(n-5)$

$=4(n^2+2n+1)-4(n^2-5n+3n-15)$

$=4n^2+8n+4-4n^2+20n-12n+60$

$D=16n+64$

1) $D>0$ $16n+64>0\Rightarrow16n>-64\Rightarrow n>-4,\quad x_1\neq x_2\in R$

2) $D=0$ $16n+64=0\Rightarrow n=-4\quad x_1=x_2\in R$

3) $D<0$ $16n+64<0\Rightarrow n<-4\quad x_1$ and x_2 are complex numbers

Example 2. Find the value for $k \in R$, that equation $kx^2 + (k+1)x + 2 = 0$ has double solution.

Solution: Must be D=0 and $a \neq 0$

$$kx^2 + (k+1)x + 2 = 0 \;\Rightarrow\; \begin{matrix} a = k \\ b = k+1 \\ c = 2 \end{matrix} \;\Rightarrow\; k \neq 0$$

$$D = b^2 - 4ac = (k+1)^2 - 4 \cdot k \cdot 2 = k^2 + 2k + 1 - 8k = k^2 - 6k + 1$$

$$D = k^2 - 6k + 1 = 0$$

$$k^2 - 6k + 1 = 0 \;\Rightarrow\; \begin{matrix} a = 1 \\ b = -6 \\ c = 1 \end{matrix}$$

$$k_{1,2} = \frac{-b \;\pm\sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \;\pm\sqrt{(-6)^2 - 4 \cdot 1 \cdot 1}}{2} = \frac{6 \pm \sqrt{32}}{2}$$

$$\sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$$

So:

$$k_{1,2} = \frac{6 \pm 4\sqrt{2}}{2} = \frac{2\left(3 \pm 2\sqrt{2}\right)}{2} = 3 \pm 2\sqrt{2}$$

$$k_1 = 3 + 2\sqrt{2}$$

$$k_2 = 3 - 2\sqrt{2}$$

Example 3. Find the value for $m \in R$, that equation $mx^2 - 4x + 1$ has real and different solutions.

Solution:

Must be $D > 0$ and $a \neq 0$

$$\begin{matrix} a = m \Rightarrow m \neq 0 \\ b = -4 \quad \Rightarrow \\ c = 1 \end{matrix}$$

$$D = b^2 - 4ac$$

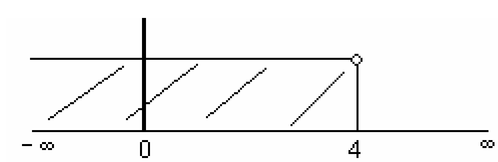
$$D = (-4)^2 - 4 \cdot m \cdot 1$$

$$D = 64 - 4m > 0$$

$$16 - 4m > 0$$

$$-4m > -16$$

$$m < 4$$



0 can not be!

So: $m \in (-\infty, 0) \cup (0, 4)$

Example 4. For what value m, equation x^2-8x+m has complex solutions?

Solution:

Must be $D<0$ i $a\neq 0$

$$\begin{aligned} a&=1\neq 0 \\ b&=-8 \\ c&=m \end{aligned} \quad\Rightarrow\quad \begin{aligned} &D=b^2-4ac \\ &D=(-8)^2-4\cdot m\cdot 1 \\ &D=64-4m<0 \\ &-4m<-64 \\ &m>16\Rightarrow m\in(16,\infty) \end{aligned}$$

Example 5. For what value $k\in R$, equation $kx^2+6x+3=0$ has no real solutions?

Solution:

When there is no real solutions, there are complex : $D<0$ and $a\neq 0$

$$kx^2+6x+3=0\Rightarrow \begin{aligned} a&=k \\ b&=6 \\ c&=3 \end{aligned} \quad \Rightarrow \quad k\neq 0$$

$$D=b^2-4ac$$

$$D=6^2-4\cdot k\cdot 3=36-12k$$

$$36-12k<0$$

$$-12k<-36$$

$$k>3\Rightarrow k\in(3,\infty)$$

Example 6. For what value $m\in R$ equation $(2m+1)x^2-(2m+1)x+2{,}5=0$ has real and different solutions?

Solution: $D>0$ and $a\neq 0$

$$\begin{aligned} a&=2m+1 \\ b&=-(2m+1) \\ c&=-2{,}5 \end{aligned} \qquad a\neq 0\Rightarrow \quad 2m+1\neq 0\Rightarrow \quad m\neq -\frac{1}{2}$$

$$D=b^2-4ac$$

$$D=[-(2m+1)]^2-4\cdot[2m+1]\cdot 2{,}5$$

$$D=(2m+1)^2-10(2m+1)$$

$$D=4m^2+4m+1-20-10$$

$$D=4m^2-16m-9>0$$

$$4m^2 - 16m - 9 = 0$$

$$a = 4 \qquad b = -16 \qquad c = -9$$

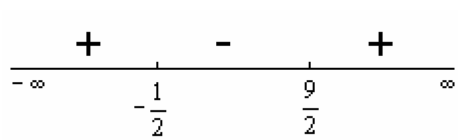
$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m_{1,2} = \frac{16 \pm \sqrt{256 - 144}}{8} = \frac{16 + 20}{8}$$

$$m_1 = \frac{36}{8} = \frac{9}{2}$$

$$m_2 = -\frac{4}{8} = -\frac{1}{2}$$

(See square inequalities):



$D > 0 \rightarrow$ choose where $+$ $\qquad m \in \left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{9}{2}, \infty\right)$